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ADDENDUM

Addendum to 'On the derivative of the Legendre function of the first kind with respect to its degree'

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Abstract

In our recent paper (Szmytkowski 2006 *J. Phys. A: Math. Gen.* **39** 15147; 2007 *J. Phys. A: Math. Theor.* **40** 7819 (corrigendum)), we have investigated the derivative of the Legendre function of the first kind, $P_\nu(z)$, with respect to its degree. In this addendum, we derive some further representations of $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$ with $n \in \mathbb{N}$. The results are used to obtain in an elementary way two formulae for the Legendre function of the second kind of integer degree, $Q_n(z)$.

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In a recent paper [1], we have extensively investigated the derivative of the Legendre function of the first kind, $P_\nu(z)$, with respect to its degree ν, with emphasis on the particular case of $\nu = n \in \mathbb{N}$. We have shown that $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$ is of the form

$$\left.\frac{\partial P_\nu(z)}{\partial \nu}\right|_{\nu=n} = P_n(z)\ln\frac{z+1}{2} + R_n(z) \qquad (n \in \mathbb{N}), \tag{1}$$

where $P_n(z)$ is the Legendre polynomial and $R_n(z)$ is a polynomial in z of degree n. Some previously known formulae for $R_n(z)$, due to Bromwich and Schelkunoff, have been rederived by us and a new expression has also been given.

After the paper [1] has been published, it has come to our attention that we have not fully exploited the Jolliffe's formula

$$\left.\frac{\partial P_\nu(z)}{\partial \nu}\right|_{\nu=n} = -P_n(z)\ln\frac{z+1}{2} + \frac{1}{2^{n-1}n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z^2-1)^n\ln\frac{z+1}{2}\right] \tag{2}$$

discussed in section 5.1 of [1]. In this addendum, it will be shown how this formula may be used to derive two further new representations of $R_n(z)$ and to rederive, in a much simpler way, the representation (5.90). We shall also obtain two expressions for the Legendre function of the second kind, $Q_n(z)$, different from those given in [1].

We begin with the observation that from (1) and (2) it follows that

$$R_n(z) = -2P_n(z)\ln\frac{z+1}{2} + \frac{1}{2^{n-1}n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z-1)^n(z+1)^n\ln\frac{z+1}{2}\right]. \tag{3}$$

Obviously, this may be rewritten as

$$R_n(z) = -2P_n(z)\ln\frac{z+1}{2} + \frac{1}{2^{n-1}n!}\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^{n-k}(z-1)^n}{\mathrm{d}z^{n-k}}\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left[(z+1)^n\ln\frac{z+1}{2}\right]. \tag{4}$$

Preparing the next step, we provide here the following differentiation formula:

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}[z^\alpha\ln z] = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)}z^{\alpha-k}\ln z + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)}[\psi(\alpha+1)-\psi(\alpha-k+1)]z^{\alpha-k}$$
$$(k\in\mathbb{N};\ \alpha\in\mathbb{C}), \tag{5}$$

where

$$\psi(\zeta) = \frac{1}{\Gamma(\zeta)}\frac{\mathrm{d}\Gamma(\zeta)}{\mathrm{d}\zeta} \tag{6}$$

is the digamma function. This formula may be easily proved by mathematical induction by using basic properties of the digamma function [2, 3]. Rewriting (5) in the form

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}[z^\alpha\ln z] = \frac{\mathrm{d}^k z^\alpha}{\mathrm{d}z^k}\ln z + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)}[\psi(\alpha+1)-\psi(\alpha-k+1)]z^{\alpha-k}$$
$$(k\in\mathbb{N};\ \alpha\in\mathbb{C}) \tag{7}$$

and using (7) in (4), after straightforward manipulations we obtain

$$R_n(z) = -2P_n(z)\ln\frac{z+1}{2} + \frac{1}{2^{n-1}n!}\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^{n-k}(z-1)^n}{\mathrm{d}z^{n-k}}\frac{\mathrm{d}^k(z+1)^n}{\mathrm{d}z^k}\ln\frac{z+1}{2}$$
$$+2\left(\frac{z+1}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}^2[\psi(n+1)-\psi(n-k+1)]\left(\frac{z-1}{z+1}\right)^k. \tag{8}$$

From the Rodrigues formula

$$P_n(z) = \frac{1}{2^n n!}\frac{\mathrm{d}^n(z^2-1)^n}{\mathrm{d}z^n} \tag{9}$$

it follows that

$$P_n(z) = \frac{1}{2^n n!}\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^{n-k}(z-1)^n}{\mathrm{d}z^{n-k}}\frac{\mathrm{d}^k(z+1)^n}{\mathrm{d}z^k}. \tag{10}$$

Hence, it is seen that the sum of the first and the second terms on the right-hand side of (8) is zero and we have

$$R_n(z) = 2\left(\frac{z+1}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}^2[\psi(n+1)-\psi(n-k+1)]\left(\frac{z-1}{z+1}\right)^k. \tag{11}$$

If the summation variable is changed from k to $k' = n-k$, use is made of one of the elementary properties of the binomial coefficient and the primes are omitted, (11) is cast into

$$R_n(z) = 2\left(\frac{z-1}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}^2[\psi(n+1)-\psi(k+1)]\left(\frac{z+1}{z-1}\right)^k. \tag{12}$$

(It is to be noted that the lower summation limit in (11) may be shifted to 1 while the upper summation limit in (12) to $n-1$.) The two representations of $R_n(z)$ given in (11) and (12) supplement formulae collected in section 5.2 in [1].

We have already mentioned that the Jolliffe's formula (2) may also be exploited to rederive in a simple manner formula (5.90) from [1]. To show this, we note that if the binomial theorem is used to expand $(z-1)^n$ in a sum of powers of $z+1$ and the result is inserted into (3), this yields

$$R_n(z) = -2P_n(z)\ln\frac{z+1}{2} + 2\sum_{k=0}^{n}\frac{(-1)^{n+k}}{2^k k!(n-k)!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z+1)^{n+k}\ln\frac{z+1}{2}\right]. \tag{13}$$

By applying formula (5), after some obvious movements the above equation becomes

$$R_n(z) = -2P_n(z)\ln\frac{z+1}{2} + 2\sum_{k=0}^{n}(-1)^{n+k}\frac{(n+k)!}{(k!)^2(n-k)!}\left(\frac{z+1}{2}\right)^k\ln\frac{z+1}{2} + 2\sum_{k=0}^{n}(-1)^{n+k}\frac{(n+k)!}{(k!)^2(n-k)!}[\psi(n+k+1)-\psi(k+1)]\left(\frac{z+1}{2}\right)^k. \tag{14}$$

By virtue of the Murphy's formula

$$P_n(z) = \sum_{k=0}^{n}(-1)^{n+k}\frac{(n+k)!}{(k!)^2(n-k)!}\left(\frac{z+1}{2}\right)^k, \tag{15}$$

we see that the first and the second terms on the right-hand side of (14) cancel out. This yields

$$R_n(z) = 2\sum_{k=0}^{n}(-1)^{n+k}\frac{(n+k)!}{(k!)^2(n-k)!}[\psi(n+k+1)-\psi(k+1)]\left(\frac{z+1}{2}\right)^k \tag{16}$$

which is identical with equation (5.90) in [1].

Having established the representations (11) and (12) of $R_n(z)$, we may obtain two expressions for the Legendre function of the second kind of integer degree, $Q_n(z)$. To show this, we recall (cf [1, sections 5.3 and 6.1]) that the Christoffel polynomial $W_{n-1}(z)$ appearing in the equation

$$Q_n(z) = \frac{1}{2}P_n(z)\ln\frac{z+1}{z-1} - W_{n-1}(z) \qquad (z\in\mathbb{C}\setminus[-1,1]) \tag{17}$$

is related to the polynomial $R_n(z)$ through

$$W_{n-1}(z) = -\tfrac{1}{2}[R_n(z)-(-1)^n R_n(-z)]. \tag{18}$$

Using $R_n(z)$ in the form (11) and employing (12) to evaluate $R_n(-z)$ gives

$$W_{n-1}(z) = \left(\frac{z+1}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}^2[\psi(n-k+1)-\psi(k+1)]\left(\frac{z-1}{z+1}\right)^k. \tag{19}$$

If the roles played by (11) and (12) are interchanged, this results in

$$W_{n-1}(z) = \left(\frac{z-1}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}^2[\psi(k+1)-\psi(n-k+1)]\left(\frac{z+1}{z-1}\right)^k. \tag{20}$$

Inserting (19) and (20) into (17) leads to the following representations of $Q_n(z)$:

$$Q_n(z) = \frac{1}{2}P_n(z)\ln\frac{z+1}{z-1} \mp \left(\frac{z\pm1}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}^2[\psi(n-k+1)-\psi(k+1)]\left(\frac{z\mp1}{z\pm1}\right)^k \qquad (z\in\mathbb{C}\setminus[-1,1]). \tag{21}$$

These representations of $Q_n(z)$ are seen to be direct counterparts of the following representations of $P_n(z)$ [4, p 17]:

$$P_n(z) = \left(\frac{z \pm 1}{2}\right)^n {}_2F_1\left(-n, -n; 1; \frac{z \mp 1}{z \pm 1}\right) = \left(\frac{z \pm 1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{z \mp 1}{z \pm 1}\right)^k. \tag{22}$$

For $z = x \in [-1, 1]$, the Legendre function of the second kind of integer degree may be expressed as

$$Q_n(x) = \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x} - W_{n-1}(x) \qquad (-1 \leqslant x \leqslant 1), \tag{23}$$

where

$$W_{n-1}(x) = W_{n-1}(z = x). \tag{24}$$

Adopting the common parametrization

$$x = \cos\theta \qquad (0 \leqslant \theta \leqslant \pi) \tag{25}$$

from (23), (24), (19) and (20) we obtain

$$\begin{aligned} Q_n(\cos\theta) = {} & P_n(\cos\theta) \ln \cot \frac{\theta}{2} \\ & - \cos^{2n} \frac{\theta}{2} \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 [\psi(n-k+1) - \psi(k+1)] \tan^{2k} \frac{\theta}{2} \end{aligned} \tag{26}$$

and

$$\begin{aligned} Q_n(\cos\theta) = {} & P_n(\cos\theta) \ln \cot \frac{\theta}{2} \\ & + \sin^{2n} \frac{\theta}{2} \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k}^2 [\psi(n-k+1) - \psi(k+1)] \cot^{2k} \frac{\theta}{2}. \end{aligned} \tag{27}$$

These representations of $Q_n(\cos\theta)$ harmonize with the following known [4, p 18] expressions for $P_n(\cos\theta)$:

$$\begin{aligned} P_n(\cos\theta) &= \cos^{2n} \frac{\theta}{2} \, {}_2F_1\left(-n, -n; 1; -\tan^2 \frac{\theta}{2}\right) \\ &= \cos^{2n} \frac{\theta}{2} \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \tan^{2k} \frac{\theta}{2} \end{aligned} \tag{28}$$

and

$$\begin{aligned} P_n(\cos\theta) &= (-1)^n \sin^{2n} \frac{\theta}{2} \, {}_2F_1\left(-n, -n; 1; -\cot^2 \frac{\theta}{2}\right) \\ &= \sin^{2n} \frac{\theta}{2} \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k}^2 \cot^{2k} \frac{\theta}{2}, \end{aligned} \tag{29}$$

respectively.

We are not aware of any appearance of either of formulae (11), (12), (21), (26) or (27) in the literature.

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